Journal of Statistical Physics, Vol. 72, Nos. 1/2, 1993

Continuous Spectrum in the Ground State of Two Spin-1/2 Models in the Infinite-Volume Limit

Martin Pokorny[1]

Received October 5, 1992

We show that in the ground states of the infinite-volume limits of both the spin-1/2 anisotropic antiferromagnetic Heisenberg model (in dimensions $d \geqslant 2$), and the ferromagnetic Ising model in a strong transverse field (in dimensions $d \geqslant 1$) there is an interval in the spectrum above the mass gap which contains a continuous band of energy levels. We use the methods of Bricmont and Fröhlich to develop our expansions, as well as a method of Kennedy and Tasaki to do the expansions in the quantum mechanical limit. Where the expansions converge, they are then shown to have spectral measures which have absolutely continuous parts on intervals above the mass gaps.

KEY WORDS: Quantum mechanical spin systems; polymer expansions; perturbative spectral analysis.

1. INTRODUCTION

In this paper, we consider the problem of analyzing the spectrum of a quantum mechanical spin system in the infinite-volume limit. Unfortunately, this requires the study of an unbounded operator in a setting where the operator is no longer defined. While the Hamiltonian is undefined in the infinite-volume limit, the states are well defined. We can define an infinite-volume thermodynamic state as a functional $\omega(\cdot)$ acting on a C^*-algebra of operators known as the *quasilocal observables*. The values of the functional are called the expectations of the observables in the state. The expectations of observables can also be defined in finite volumes, and we take the limits of the finite-volume expectations to get their infinite-volume

[1] Program in Applied Mathematics, University of Arizona, Tucson, Arizona.

0022-4715/93/0700-0381$07.00/0 © 1993 Plenum Publishing Corporation

values. A ground state is a functional ω which has the following property for every quasilocal observable $\mathcal{O}$ in the algebra:

$$\omega(\mathcal{O}^*[H, \mathcal{O}]) \geqslant 0 \tag{1}$$

See refs. 2 and 3 for details. Typically, once we have an expression for the infinite-volume expectation at any temperature, we can find the expectation in the ground state by letting the temperature go to zero.

We usually can study the time evolution of an expectation by using the time-dependent operator $e^{itH}\mathcal{O}e^{-itH}$, where $\mathcal{O}$ is the observable and H is the Hamiltonian. Although H is not defined in the infinite-volume limit, for real values of t, the operator $e^{itH}\mathcal{O}e^{-itH}$ is well defined.[3] To get some information about the spectrum in the ground state we could calculate the quantity $\omega(\mathcal{O}^* e^{itH}\mathcal{O}e^{-itH})$. However, for the expansions in our calculations to work we must calculate the above in purely imaginary t. Unfortunately in imaginary t the quantity has the disadvantage of not being defined for all temperatures. What makes the method work in this case is that the system is in the ground state, requiring the limit $\beta \to \infty$ (with β equal to the inverse temperature). When we evolve the observable in imaginary time we can use the usual approach to find an expectation, first rigorously done in ref. 6, of writing a quantum mechanical spin system as a classical spin system in one additional dimension. Nonetheless, the quantity $\omega(\mathcal{O}^* e^{-tH}\mathcal{O}e^{tH})$ can yield information about the spectrum. If the preceding quantity could be calculated for all observables in the algebra for all t, we could regain the complete spectrum. Unfortunately, that is not always possible. In this paper, however, we do perform this procedure for one observable and large t, and thus can obtain some information about the spectrum of H near the ground-state energy.

The approach which we take to this problem for the two models described in this paper will be to write all of the quantities which we need in a finite volume and at finite β, and then proceed to the infinite-volume, infinite-β limits. The expansions we develop are based on the work done by Bricmont and Fröhlich[4] on classical spin systems, except that in our case one of the dimensions is continuous. The two models which appear in the paper are the spin-1/2 anisotropic antiferromagnetic Heisenberg model in dimensions greater than or equal to two, and the ferromagnetic Ising model in a strong transverse field in dimensions greater than or equal to one. We will show that in the infinite-volume ground state of both of these models, the spectrum above the lowest energy level contains a continuous part and is separated from the lowest level by an energy gap. The existence of an energy gap has been shown previously.[9,13,12,1,8] In a finite volume, the energy level above the lowest level is N-fold degenerate in both unperturbed models. What we have shown is that under the perturbation, in the infinite-volume limit, these levels spread into a continuous band of energy levels.

The only observable which is used in our calculations is σ_0^x. For the case of the antiferromagnetic Heisenberg model, in the path-space formulation of the quantity discussed in the second paragraph above, σ_0^x "flips" the spin at the origin at times 0 and t, and we find that the excitations which we can control consist of deviations of a simple "tube" of plaquettes which connects the origin between "times" 0 and t (included in this group of excitations are all sites not in the ground state which exist apart from the "tube"). The energy penalty to excite this tube, i.e., to make it include at any single time n sites with $n > 1$, will roughly speaking scale like $n^{(d-1)/d}$. This energy penalty is enough for us to control these lowest energy excitations in our calculation. It also shows why the calculation which we present fails for $d = 1$; that is, there is no energy penalty due to increasing "surface" as the volume n of the excitation grows. In the case of the Ising model in a strong transverse field, the path-space formulation does not result in plaquettes surrounding sites, but rather in paths which simply connect sites along the "bonds." Excitations are again defined in a similar manner, but here every path segment carries an energy penalty. Thus, even in one dimension, there is an increasing energy penalty for each excitation as its volume increases.

Regarding our proofs of the existence of continuous spectrum, various exactly solvable models have previously been shown to have continuous spectrum,[11] but only in one dimension. Both of our results, though perturbative in nature, are valid for dimensions greater than (and, in one case, equal to) one. We cannot rule out bound states within the continuum, because we can only do the calculations with the single observable σ_0^x; and to get all of the spectrum would require doing the calculations with all observables for all values of t. We comment on the effect which other observables might have on the spectrum, as well as the difficulties encountered in getting a result with other observables, in the last section of the paper.

2. MAIN RESULTS

2.1. Statement of the Theorem

In this paper we primarily consider the anisotropic antiferromagnetic spin-1/2 Heisenberg model. We define the model by giving the Hamiltonian of our system as well as the Hilbert space on which the Hamiltonian acts.

The Hilbert space is the tensor product space of $|\Lambda|$ copies of $\mathbf{C}^2$, and the Hamiltonian acting on this Hilbert space is

$$H_\Lambda = \sum_{\langle ij \rangle} \frac{\varepsilon}{2} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) + \frac{1}{2} (\sigma_i^z \sigma_j^z + 1) \tag{2}$$

where the sum is over nearest neighbor pairs of sites $\langle ij \rangle$ on the lattice Λ with periodic boundary conditions, and the σ's are the usual Pauli spin operators. The lattice Λ will be a subset of $\mathbf{Z}^d$. To investigate the structure of the spectrum we will be calculating the quantity $\mathrm{Tr}(\sigma_0^x e^{-tH_\Lambda} \sigma_0^x e^{-(\beta-t)H_\Lambda})$ as a function of t. As a preliminary step we apply a unitary transformation which rotates every spin on the "odd" sublattice of Λ about the y axis by π radians. After this transformation (2) becomes

$$H_\Lambda = -\sum_{\langle ij \rangle} \varepsilon(\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-) + \tfrac{1}{2}(\sigma_i^z \sigma_j^z - 1) \tag{3}$$

where $\sigma_i^+ = (\sigma_i^x + i\sigma_i^y)/2$ and $\sigma_i^- = (\sigma_i^x - i\sigma_i^y)/2$. The operator σ_i^+ acting on a "down" spin changes it into an "up" spin, while an "up" spin is in the operator's kernel. σ_i^- has a similar action except with "down" and "up" interchanged.

We will also consider the Ising model in a strong transverse field. The Hamiltonian for this model is

$$H_\Lambda = -\varepsilon \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z + \tfrac{1}{2} \sum_i (\sigma_i^x + 1) \tag{4}$$

In this model, too, a preliminary unitary transformation will make the model more amenable to the following expansions. The transformation in this case consists of a rotation of every site about the y axis by $\pi/2$ radians. The result of this transformation on (4) is

$$H_\Lambda = -\tfrac{1}{2} \sum_i (\sigma_i^z - 1) - \varepsilon \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x \tag{5}$$

In the following we will do the calculations for the anisotropic Heisenberg model in detail, and will merely indicate at which points a similar calculation for the strong-field Ising model differs significantly from the one given.

We now state the theorem which we will prove in this paper.

Theorem 1. For Hamiltonians given in (2) (in dimensions $d \geqslant 2$) and (4) (in dimensions $d \geqslant 1$), define the following function in finite volume Λ and at finite inverse temperature β:

$$f_\beta^\Lambda(t) = \frac{\mathrm{Tr}(\sigma_0^x e^{-tH_\Lambda} \sigma_0^x e^{-(\beta-t)H_\Lambda})}{\mathrm{Tr}(e^{-\beta H_\Lambda})} \tag{6}$$

Then there is an $\varepsilon_0 > 0$ such that when $\varepsilon < \varepsilon_0$, the function $f_\beta^A(t)$ converges to a function $f(t)$ as $\beta \to \infty$ and subsequently $|\Lambda| \to \infty$. In addition, there exists a unique Baire measure such that

$$f(t) = \int_0^\infty \exp(-tE)\, d\mu(E) \tag{7}$$

and there exist $m_0 > 0$ which depends analytically on ε, and m_1 which is greater than m_0, such that $d\mu$ has no support between 0 and m_0, yet has support throughout the entire interval $[m_0, m_1)$. On this interval $d\mu$ is absolutely continuous with respect to Lebesgue measure. The width of the interval is asymptotically $O(\varepsilon)$ as $\varepsilon \to 0$.

2.2. The First Expansion

Using a slight variant of the Trotter product formula, we begin the first expansion of $\mathrm{Tr}(\sigma_0^x e^{-tH_\Lambda} \sigma_0^x e^{-(\beta - t)H_\Lambda})$ for (3):

$$\begin{aligned}
&\mathrm{Tr}(\sigma_0^x \exp(-tH_\Lambda)\, \sigma_0^x \exp[-(\beta - t)\, H_\Lambda])\\
&\quad = \lim_{N \to \infty} \mathrm{Tr}\Bigg(\sigma_0^x \Bigg\{\exp\Bigg(\sum_{\langle ij\rangle} \frac{1}{2N}(\sigma_i^z \sigma_j^z - 1)\Bigg)\\
&\qquad \times \Bigg[1 + \frac{\varepsilon}{N} \sum_{\langle ij\rangle} (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-)\Bigg]\Bigg\}^{Nt}\\
&\qquad \times \sigma_0^x \Bigg\{\exp\Bigg(\sum_{\langle ij\rangle} \frac{1}{2N}(\sigma_i^z \sigma_j^z - 1)\Bigg)\\
&\qquad \times \Bigg[1 + \frac{\varepsilon}{N} \sum_{\langle ij\rangle} (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-)\Bigg]\Bigg\}^{N(\beta - t)}\Bigg)
\end{aligned} \tag{8}$$

We choose the usual complete set of basis vectors $\{|\Psi\rangle\}$, where $\sigma_i^z |\Psi\rangle = \sigma_i |\Psi\rangle$ with $\sigma_i = \pm 1$ (if $\sigma_i = +1$, the spin at site i is "up"; if $\sigma_i = -1$, the spin at site i is "down") for each $i \in \Lambda$. By inserting a sum over the set of basis vectors between each of the $N\beta + 2$ factors in the product, the right-hand side of (8) becomes

$$\begin{aligned}
&\sum_{\Psi_k} \langle \Psi_0 |\, \sigma_0^x \,| \Psi_{1/N}\rangle \langle \Psi_{t+1/N} |\, \sigma_0^x \,| \Psi_{t+2/N}\rangle\\
&\quad \times \prod_{T = 1/N}^{t} \langle \Psi_T | \exp\Bigg(\sum_{\langle ij\rangle} \frac{1}{2N}(\sigma_i^z \sigma_j^z - 1)\Bigg)\\
&\quad \times \Bigg[1 + \frac{\varepsilon}{N} \sum_{\langle ij\rangle} (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-)\Bigg] | \Psi_{T+1/N}\rangle
\end{aligned}$$

$$\times \prod_{T=t+2/N}^{\beta+1/N} \langle \Psi_T | \exp\left(\sum_{\langle ij \rangle} \frac{1}{2N} (\sigma_i^z \sigma_j^z - 1) \right)$$

$$\times \left[1 + \frac{\varepsilon}{N} \sum_{\langle ij \rangle} (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-) \right] | \Psi_{T+1/N} \rangle \tag{9}$$

with $|\Psi_{\beta+2/N}\rangle \overset{\text{def}}{=} |\Psi_0\rangle$. The exponential terms are diagonal in the basis, so we now have

$$\sum_{\Psi_k} \langle \Psi_0 | \sigma_0^x | \Psi_{1/N} \rangle \langle \Psi_{t+1/N} | \sigma_0^x | \Psi_{t+2/N} \rangle$$

$$\times \prod_{T=1/N}^{t} \exp\left(\sum_{\langle ij \rangle} \frac{1}{2N} (\sigma_i(T)\, \sigma_j(T) - 1) \right)$$

$$\times \langle \Psi_T | \, 1 + \frac{\varepsilon}{N} \sum_{\langle ij \rangle} (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-) \, | \Psi_{T+1/N} \rangle$$

$$\times \prod_{T=t+2/N}^{\beta+1/N} \exp\left(\sum_{\langle ij \rangle} \frac{1}{2N} (\sigma_i(T)\, \sigma_j(T) - 1) \right)$$

$$\times \langle \Psi_T | \, 1 + \frac{\varepsilon}{N} \sum_{\langle ij \rangle} (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-) \, | \Psi_{T+1/N} \rangle \tag{10}$$

where $\sigma_i(k)$ is the value of the spin at site i at $T=k$.

We will denote by Ξ the time axis ($=\{0, 1/N, ..., \beta + 1/N\}$). Each term in the sum (10) has a geometrical interpretation in terms of *plaquettes* in $\Lambda \times \Xi$. When we consider the unit cubes centered on the sites in a d-dimensional cubic (or rectangular) lattice, then we define the plaquettes to be the $(d-1)$-dimensional faces of those unit cubes. We say that a particular plaquette is *dual* to the bond in the lattice which is bisected by the plaquette. Consider the plaquettes dual to the bonds between two sites with opposite spins. Such plaquettes dual to bonds in Λ^* (which is the set of bonds in the lattice Λ) will be called *time plaquettes*, while such plaquettes dual to bonds in the Ξ direction will be called *space plaquettes*. For the strong-field Ising model the geometric interpretation is quite different. In this case the spins not in the ground state will appear to hop from site to site, and a term in the sum over configurations will be a collection of bonds in $(\Lambda \times \Xi)^*$ corresponding to the "paths" of the spins out of the ground state. Bonds parallel to the Ξ axis will be called *time bonds*, and bonds in Λ^* will be called *space bonds*.

Each time plaquette will occur in (10) with a factor of $\exp(-1/N)$. The off-diagonal factors in (10) are given by

$$\langle \Psi_k |\ 1 + \frac{\varepsilon}{N} \sum_{\langle ij \rangle} (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-)\ | \Psi_{k+1/N} \rangle$$
$$= \begin{cases} 1 & \text{if} \quad \Psi_k = \Psi_{k+1/N} \\ \dfrac{\varepsilon}{N} & \text{if} \quad \langle \Psi_k |\ \sigma_i^+ \sigma_j^+\ | \Psi_{k+1/N} \rangle = 1 \text{ for a single } \langle ij \rangle \in \Lambda^* \\ \dfrac{\varepsilon}{N} & \text{if} \quad \langle \Psi_k |\ \sigma_i^- \sigma_j^-\ | \Psi_{k+1/N} \rangle = 1 \text{ for a single } \langle ij \rangle \in \Lambda^* \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Thus, the space plaquettes arising from the off-diagonal terms will occur in (10) in connected pairs (which we will call *space rectangles*) with factors of ε/N. As usual, in each term in the sum every edge between plaquettes must have an even number of plaquettes touching it. We call any such connected set of plaquettes a *contour*. Thus we can associate to each contour ω a *weight* $W(\omega)$ composed of a factor of $\exp(-1/N)$ for every time plaquette in ω, and a factor of ε/N for every space rectangle in ω. The contours in the strong-field Ising model will be connected sets of bonds, and the weights of time and space bonds are also $\exp(-1/N)$ and ε/N, respectively.

In terms of contours and their weights, (10) is now a weighted sum over sets of contours in $\Lambda \times \Xi$:

$$\mathrm{Tr}(\sigma_0^x e^{-tH_\Lambda} \sigma_0^x e^{-(\beta - t)H_\Lambda}) = \lim_{N \to \infty} \tilde{\sum_{\Omega}} W(\Omega) \tag{12}$$

where $\tilde{\sum}$ is a sum over allowed unconnected sets of contours in $\Lambda \times \Xi$, and $W(\Omega)$ is the total weight of the contours in Ω. The allowed sets of contours which occur in $\tilde{\sum}$ are those for which (1) at most only one space rectangle exists at each T value, (2) there is a single space plaquette at the origin at both $T=0$ and $T=t+1/N$, and (3) every closed path in $\Lambda \times \Xi$ crosses an even number of plaquettes. An example contour set is shown in Fig. 1. The third constraint does not arise in the strong-field Ising model because there are no plaquettes in that model, which makes the Ising model somewhat easier to calculate. The second constraint will be replaced in the Ising model by a requirement that there is a path which originates at the origin at $T=0$ and ends at the origin at $T=t+1/N$.

Returning to the Heisenberg model, the first constraint on the sum can be dropped with the introduction of a correction vanishing as N increases.[7] The third constraint derives from the periodicity of $\Lambda \times \Xi$ in every direction, the periodicity in the space directions deriving from the boundary conditions, and the periodicity in the time direction deriving from the trace in (8). Given the periodicity, any closed path in $\Lambda \times \Xi$ must

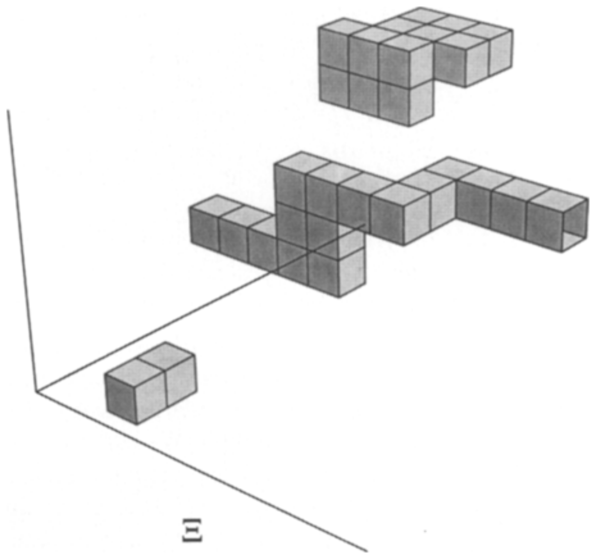

Fig. 1. An example of a set of allowed contours ($d=2$), including a portion of the contour which runs from $(0, 0)$ to $(0, t+1/N)$.

cross an even number of spin flips—hence the third constraint. We must also develop an expansion of $\mathrm{Tr}(e^{-\beta H_\Lambda})$, but this expansion will be very similar to (12) except without the second constraint.

The function we now need to investigate is

$$f^{\Lambda}_{\beta}(t) = \lim_{N\to\infty} \frac{\mathrm{Tr}(\sigma^x_0 e^{-tH_\Lambda} \sigma^x_0 e^{-(\beta-t)H_\Lambda})}{\mathrm{Tr}(e^{-\beta H_\Lambda})} \tag{13}$$

To proceed we need to first show that expansions exist for both the numerator and the denominator of $f^{\Lambda}_{\beta}(t)$ in the infinite-volume and zero-temperature limits. We will formulate our expansions in the infinite-N limit as done in ref. 8. In the $N\to\infty$ limit, $\Xi=[0, \beta]$ with periodic boundary conditions. We will now consider the Ξ direction to be blocked, or divided up, into partitions of length τ. In the blocked picture we introduce new types of contours which we name *polymers*. The polymers will consist of time plaquettes, and blocks centered at sites in Λ with length in the Ξ direction equal to τ. Specifically a block is a cube in $\Lambda\times\Xi$ of the following type:

$$B(i_1,..., i_d, m) = [i_1-\tfrac{1}{2}, i_1+\tfrac{1}{2})\times \cdots \times [i_d-\tfrac{1}{2}, i_d+\tfrac{1}{2})[m\tau, (m+1)\tau) \tag{14}$$

for $(i_1,..., i_d) \in \Lambda$. {In the Ising model, the geometrical objects in the blocked picture will be time bonds whose lengths are multiples of τ, and two-dimensional rectangles $[i, i+1) \times [m\tau, (m+1)\tau)$.} Let $\mathscr{S}(\omega)$ be the set of blocks in the blocked picture which intersect any space rectangle of ω, and let $\mathscr{T}(\omega)$ be the set of time plaquettes in the blocked picture for which the spins on opposite sides of the plaquette are never aligned (throughout the length of the plaquette).

Time plaquettes and blocks in the blocked picture are considered connected if they touch anywhere. Fix a connected set γ of plaquettes and blocks in the blocked picture. For any polymer ω with $\mathscr{S}(\omega) \cup \mathscr{T}(\omega) = \gamma$, it is easy to see that

$$\prod_T \exp\left\{\frac{1}{2N} \sum [\sigma_i(T)\,\sigma_j(T) - 1]\right\} \leqslant \exp[-|\mathscr{T}(\omega)|\,\tau] \tag{15}$$

The weights for the blocks in $\mathscr{S}$ can be bounded by use of a *comparison Hamiltonian*. This method is developed in ref. 8, but we reproduce it here for completeness. If we let m be the number of space rectangles in ω, then

$$\begin{aligned} &\prod_T \langle \Psi_T | \frac{\varepsilon}{N} \sum_{\langle ij \rangle} (\sigma_i^+\sigma_j^+ + \sigma_i^-\sigma_j^-)\,|\Psi_{T+1/N}\rangle \\ &\quad = (\varepsilon\tau)^m \prod_T \langle \Psi_T | \frac{1}{\tau N} \sum_{\langle ij \rangle} (\sigma_i^+\sigma_j^+ + \sigma_i^-\sigma_j^-)\,|\Psi_{T+1/N}\rangle \end{aligned} \tag{16}$$

We can bound m from below by $|\mathscr{S}(\omega)|$. The comparison Hamiltonian will be

$$\hat{H}_\Lambda = -\frac{1}{\tau} \sum_{\langle ij \rangle} (\sigma_i^+\sigma_j^+ + \sigma_i^-\sigma_j^-) \tag{17}$$

If we let $\tau' = \min(\tau, -\ln(\varepsilon\tau))$, then

$$|W(\omega)| \leqslant e^{-\tau'|\gamma|}\,|\hat{W}(\omega)| \tag{18}$$

with $\hat{W}(\omega)$ equal to the weight of the polymer under the comparison Hamiltonian. We will soon show that $\sum_{\omega:\mathscr{S}(\omega)\cup\mathscr{T}(\omega)=\gamma} \hat{W}(\omega)$ is bounded above by $e^{k|\gamma|}$, with k independent of ε and τ. Therefore, given ε, we can choose τ such that $e^{-\tau}/\tau \leqslant \varepsilon < e^{-k}/\tau$, and then we have an exponentially decreasing bound

$$\left| \sum_{\omega:\mathscr{S}(\omega)\cup\mathscr{T}(\omega)=\gamma} W(\omega) \right| \leqslant e^{-n|\gamma|} \tag{19}$$

where $\eta = \tau' - k$. Therefore by standard results on polymer expansions, all the desired sums are absolutely convergent.[5,10]

To prove the claimed bound on $\hat{W}(\omega)$, let I be a set of sites, and let $\Pi(I)$ be the projection onto the states which are in the ground state off of I. Also let $I(\gamma) = \{i: B(i, 0) \in \gamma\}$, and $I_m(\gamma) = \{i: B(i, m-1) \in \gamma\}$. Next break up $\hat{H}_\Lambda$ into its components in each of the (blocked) Ξ slices:

$$\hat{H}_\Lambda^m = -\frac{1}{\tau} \sum_{\langle ij \rangle \in I_m(\gamma)} (\sigma_i^+ \sigma_j^+ + \sigma_i^- \sigma_j^-)$$

The following bound holds because every term in the expansion of the right-hand side is positive, and every term on the left-hand side occurs in the expansion of the right:

$$\sum_{\omega: \mathscr{S}(\omega) \cup \mathscr{T}(\omega) = \gamma} \hat{W}(\omega) \leqslant \operatorname{Tr}[\Pi(I(\gamma)) \exp(-\hat{H}_\Lambda^1) \exp(-\hat{H}_\Lambda^2) \cdots \exp(-\hat{H}_\Lambda^{\beta/\tau})] \tag{20}$$

It is easy to show that if A and B are both self-adjoint operators and A is also a positive operator, then

$$\operatorname{Tr}(AB) \leqslant \|B\| \operatorname{Tr}(A) \tag{21}$$

To use the above identity in the current context, we let $A = \Pi(I(\gamma))$ and $B = \exp(-\hat{H}_\Lambda^1) \cdots \exp(-\hat{H}_\Lambda^{\beta/\tau})$. We also define $[I^C(\gamma)]$ to be the number of connected components off of $I(\gamma)$. Then

$$\operatorname{Tr}[\Pi(I(\gamma))] = 2^{|I(\gamma)| + [I^C(\gamma)]} \tag{22}$$

and

$$\|\exp(-\hat{H}_\Lambda^1) \cdots \exp(-\hat{H}_\Lambda^{\beta/\tau})\| \leqslant \exp\left(\sum_n \|\hat{H}_\Lambda^n\|\right) \tag{23}$$

Because $[I^C(\gamma)] \leqslant |I(\gamma)|$ and $\|\hat{H}_\Lambda^n\| \leqslant |I_n(\gamma)|$, the claim is proved.

Standard results in polymer expansion theory now let us conclude that there is an expansion for both numerator and denominator in Eq. (13) in the blocked picture. It is now appropriate to consider the effect of the periodicity of $\Lambda \times \Xi$ on our expansions. Let us define a *sheet* as a polymer with at least one direction in $\Lambda \times \Xi$ such that all closed paths in the given direction cross the polymer at least once. The weights of sheet polymers cannot simply be the sum of contour weights over all contours which have their support on the sheet, because of the global constraint imposed by periodic boundary conditions. We define $U(\cdot)$ and $\tilde{U}(\cdot)$ to be the weights of polymers without and with this constraint, respectively. We will show, however, that up to exponentially small corrections in the size of $\Lambda \times \Xi$, we can approximate $\tilde{U}(\cdot)$ by $U(\cdot)$. As usual, we partition a set of polymers Γ into its connected components $\gamma_1,..., \gamma_n$. The results of the expansion (without the global constraint) yield

$$\sum_{\Gamma:u} \prod_i U(\gamma_i) = \exp\left[\sum_\Gamma V^C(\gamma_1,..., \gamma_n) \prod_i U(\gamma_i)\right]$$

$$\stackrel{\text{def}}{=} \exp\left[\sum_\Gamma \phi(\Gamma)\right] \tag{24}$$

where $\Gamma: u$ is a sum over sets of unconnected polymers. The definition of the connected part of the "hard-core" potential $V^C(\cdot)$ is standard, and can be found in ref. 5. At present the most important feature of $V^C(\gamma_1,..., \gamma_n)$ is that it vanishes unless $\bigcup_i \gamma_i$ is connected. A similar expression holds for the sum over sets of polymers with the global constraint. We can now show that the contribution of sheets to our expansions is exponentially small in the size of $\Lambda \times \Xi$. (Note that in the Ising model we do not have to consider any sheets or their attendant complexities.)

We must now bound the difference in the weights of polymers with and without the global constraint. This difference appears in the expansion in a term of the following form:

$$\left|\exp\left[\sum_\Gamma \tilde{\phi}(\Gamma) - \sum_\Gamma \phi(\Gamma)\right]\right| \tag{25}$$

The exponent of this expression in terms of $U(\gamma)$ is the following:

$$\sum_\Gamma \tilde{\phi}(\Gamma) - \sum_\Gamma \phi(\Gamma)$$

$$= \sum_\Gamma V^C(\gamma_1,..., \gamma_n)\left[\prod_i \tilde{U}(\gamma_i) - \prod_i U(\gamma_i)\right] \tag{26}$$

$$= \sum_{\Gamma:\ \Gamma \text{ contains a sheet}} V^C(\gamma_1,..., \gamma_n)\left[\prod_i \tilde{U}(\gamma_i) - \prod_i U(\gamma_i)\right] \tag{27}$$

Because we already have a bound on $U(\cdot)$ and $\tilde{U}(\cdot)$, and every term in the sum has a sheet of size at least $[\text{diam}(\Lambda \times \Xi)]^d$, standard results in polymer expansions allow us to conclude that there is some $\mu > 0$ such that for large enough $\text{diam}(\Lambda \times \Xi)$,

$$\left|\exp\left[\sum_\Gamma \tilde{\phi}(\Gamma) - \sum_\Gamma \phi(\Gamma)\right]\right| \leqslant O(\exp(-\mu\zeta^d)) \tag{28}$$

with $\zeta \stackrel{\text{def}}{=} \text{diam}(\Lambda \times \Xi)$.

Up to this point we have ignored the contour which includes the spin flip at the origin at $T = 0$ and $T = t$. To take it into account, we factor out the sum over this single contour (denoted by a subscript 0) in the numerator of (13). The sum over the rest of the contours will then have the restriction that no contour is connected to the contour ω_0. In the blocked picture this restriction will still hold. The sum over the rest of the polymers will still have an expansion. Therefore, for large enough ζ, (13) will become

$$\begin{aligned} f_\beta^A(t) &= \sum_{\gamma_0} U(\gamma_0) \frac{\exp(\sum_{\Gamma:\, \Gamma \cap \gamma_0 = \varnothing} \phi(\Gamma))[1 + O(\exp(-\mu\zeta^d))]}{\exp[\sum_\Gamma \phi(\Gamma)][1 + O(\exp(-\mu\zeta^d))]} \\ &= \sum_{\gamma_0} U(\gamma_0) \exp\left[\sum_{\Gamma:\, \Gamma \cap \gamma_0 \neq \varnothing} -\phi(\Gamma)\right] \frac{1 + O(\exp(-\mu\zeta^d))}{1 + O(\exp(-\mu\zeta^d))} \end{aligned} \tag{29}$$

2.3. The Second Expansion

Our next step is to formulate a second expansion based on the previous expansion (29), following the method developed in ref. 4. We must consider the lowest energy polymers which enter into our first expansion. We will then find that the partition of polymers into lowest energy polymers and higher energy polymers can be characterized by the polymers' projections on the Ξ axis. Based on this characterization, we can rewrite (29) as a sum over intervals on the Ξ axis. We can make estimates to show that this new one-dimensional expansion also exists, and at that point we will be close to finishing the proof of Theorem 1.

Consider the blocks and plaquettes in a polymer γ. Any component of a polymer which consists of only a tube of plaquettes surrounding a single site we will call a *simple tube*. Note that only γ_0 can include any simple tubes. Let $\Pi(\gamma)$ be the projection of any polymer γ onto the Ξ axis. This projection will consist of a number of intervals of length τ. If for a fixed, single τ interval $X \subset \Pi(\gamma)$ the set $\Pi^{-1}(X) \cap \gamma$ contains at least one block or a tube which is not simple, then we will say that X is in the set of *excitation intervals* of γ. Note that the projection of every polymer which is not γ_0 is in the set of excitation intervals. A typical excitation is shown in Fig. 2. In the Ising model, an interval X is in the set of excitation intervals if $\Pi^{-1}(X) \cap \gamma$ contains at least one rectangle or more than one time bond.

We now would like to develop our expansion in terms of intervals on the Ξ axis. For a fixed γ_0 we define for any connected set of τ intervals X

$$K(X, \gamma_0) = \sum_{\substack{\Gamma:\, \Gamma \cap \gamma_0 \neq \varnothing \\ \Pi(\Gamma) = X}} -\phi(\Gamma) \tag{30}$$

where $\phi(\Gamma)$ is the same function defined in (24). We can easily show that $K(\cdot, \gamma_0)$ is bounded by

$$|K(X, \gamma_0)| \leqslant c \exp(-2d\,|X|\,\eta)\,|\gamma_0 \cap \Pi^{-1}(X)| \tag{31}$$

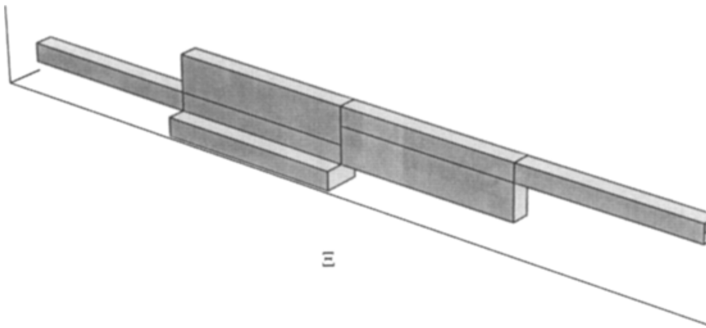

Fig. 2. A typical excitation with simple tubes at both ends ($d=2$).

where η is given in (19). Because $K(X, \gamma_0)$ is small, we will define $k(X, \gamma_0)$ by

$$\exp K(X, \gamma_0) = 1 + k(X, \gamma_0) \tag{32}$$

Those elements of γ_0 which have their projections in the excitation intervals of γ_0 we call the *excitations* of γ_0, and we denote the excitations of γ_0 by $E(\gamma_0)$. In terms of these new definitions, (29) becomes (ignoring finite-volume corrections)

$$\begin{aligned} f_\beta^A(t) &= \sum_{\gamma_0} U(\gamma_0) \prod_X [1 + k(X, \gamma_0)] \\ &= \sum_{\gamma_0} U(\gamma_0) \sum_{X_1,\ldots, X_n} \prod_i k(X_i, \gamma_0) \\ &= \sum_Y \sum_{\gamma_0 : \Pi(E(\gamma_0)) = Y} \sum_{X_1,\ldots, X_n} U(\gamma_0) \prod_i k(X_i, \gamma_0) \end{aligned} \tag{33}$$

where the sum over $X_1,\ldots, X_n$ is over distinct τ intervals which can overlap, and the sum over Y is over sets of disjoint τ intervals. The excitation interval of γ_0 will consist of disjoint intervals $Y_1,\ldots, Y_n$. Those parts of γ_0 which are not excitations consist of simple tubes. Define the left (resp. right) side of an excitation interval to be end of the interval with the smaller (larger) coordinate on the Ξ axis. An excitation of γ_0, or $\gamma_0 \cap \Pi^{-1}(Y_i)$ (for a fixed i), can have (at most) a single simple tube entering it from the left and a single simple tube exiting it from the right. If the excitation contains (0, 0) [resp. (0, t)], then the excitation will not have a tube on the left (right); otherwise the excitation will have one simple tube on both the left and the right. For the ith excitation, denote the Λ coordinates of the center of the simple tube on the left by $\mathbf{r}_i^-$, and for the simple tube on the right, $\mathbf{r}_i^+$. Clearly $\mathbf{r}_i^+ = \mathbf{r}_{i+1}^-$. If the first excitation interval includes 0, define $\mathbf{r}_1^- = 0$, and if the last excitation interval includes t, define $\mathbf{r}_n^+ = 0$. For each excitation interval define $\mathbf{r}_i = \mathbf{r}_i^+ - \mathbf{r}_i^-$. Because the tube begins and ends at $0 \in \Lambda$, we have the constraint $\delta(\sum_i \mathbf{r}_i) = 1$, where

$$\delta\left(\sum_i \mathbf{r}_i\right) = \begin{cases} 1 & \text{if } \sum_i \mathbf{r}_i = 0 \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

Between excitations, the simple tubes have a weight of $\exp[-2d(\text{length})]$. If we factor out the weight of simple tube from (0, 0) to (0, t), then we must put back in a factor of $\exp(2d\,|Y \cap [0, t)|)$ for every excitation Y. Now (33) can be rewritten as

$$f_\beta^\Lambda(t) = \exp(-2dt) \sum_{Y_1,\ldots,Y_m} \prod_i \exp(2d\,|Y_i \cap [0, t)|)$$

$$\times \sum_{\gamma_0:\, \Pi(E(\gamma_0)) = \bigcup_i Y_i} \;\sum_{X_1,\ldots,X_n} \left[\prod_i U(\Pi^{-1}(Y_i) \cap \gamma_0)\right]\left[\prod_j k(X_j, \gamma_0)\right] \tag{35}$$

where the Y_i are disjoint intervals, the X_i are distinct intervals [as in (33)]. When we describe the polymer γ_0 in terms of the $\mathbf{r}_i$ we get the following equation:

$$f_\beta^\Lambda(t) = \exp(-2dt) \sum_{Y_1,\ldots,Y_m} \prod_i \exp(2d\,|Y_i \cap [0, t)|)$$

$$\times \sum_{\mathbf{r}_1,\ldots,\mathbf{r}_m} \delta\left(\sum_i \mathbf{r}_i\right) \sum_{\substack{\gamma_0:\,\mathbf{r} \\ \Pi(E(\gamma_0)) = \bigcup_i Y_i}} \prod_i U(\Pi^{-1}(Y_i) \cap \gamma_0)$$

$$\times \sum_{X_1,\ldots,X_n} \prod_j k(X_j, \gamma_0) \tag{36}$$

where the sum over $\gamma_0 : \mathbf{r}$ means the sum over all γ_0 polymers which obey the $\mathbf{r}_i^\pm$ restrictions.

The union of all the intervals in the above equation can be broken up into disjoint intervals $I_1,\ldots, I_m$. The weights of these intervals are now

$$\mathscr{U}(I, \mathbf{r}) = \exp(2d\,|I \cap [0, t)|) \sum_{\gamma_0} \sum_{X_1,\ldots,X_n} \prod_j k(X_j, \gamma_0) \tag{37}$$

where the sums are over γ_0 and the X_i with $\Pi(E(\gamma_0)) \cup (\bigcup_i X_i) = I$, the X_i distinct, γ_0 entering the excitation as a simple tube at 0, and exiting the excitation as a simple tube at $(\mathbf{r}, |I|)$. With this definition, the expansion becomes

$$f_\beta^A(t) = \exp(-2dt) \sum_{\substack{I_1, ..., I_m: \\ \text{disjoint}}} \sum_{\mathbf{r}_1, ..., \mathbf{r}_m} \delta\left(\sum_i \mathbf{r}^i\right) \prod_i \mathscr{U}(I_i, \mathbf{r}_i) \tag{38}$$

We next want to show that there is an expansion for the sum over I's and $\mathbf{r}$'s (subject to the constraints) of the product of the weights $\mathscr{U}(I, \mathbf{r})$. To do this, we can show the appropriate bound on $|\mathscr{U}(I, \mathbf{r})|$. From (31) each factor of $k(\mathbf{X}, \gamma_0)$ in (37) is no larger than $\exp(-2d\,|X|\,\eta)$ (times a term proportional to the size of the part of γ_0 whose projection lies in X). Because γ_0 extends at least across $I \cap [0, t)$ and also must traverse a distance of $|\mathbf{r}|$ in Λ, γ_0 contributes a factor whose magnitude is bounded above by $\exp\{-[2d\,|I \cap [0, t)| + |\mathbf{r}|/2]\,\eta\}$. In addition, there are small factors that are also present because there must be an excitation throughout $|I|$. Since every X_i is an excitation already, the space between the X_i in I must be "filled" by excitations of γ_0. The excitations of γ_0 will add an additional $2(d-1)\,|I| - \sum_i |X_i|$ small factors. (Note that it is essential that $d \geqslant 2$, or this estimate will not work. In the Ising model the estimates are somewhat simpler and will also work for $d = 1$). Therefore we have the bound

$$|\mathscr{U}(I, \mathbf{r})| \leqslant O(\exp\{-[|\mathbf{r}|/2 + 2(d-1)\,|I|]\,\eta\}) \tag{39}$$

We use the following identity to transform the term $\delta(\sum_i \mathbf{r}_i)$:

$$\delta\left(\sum_i \mathbf{r}_i\right) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \exp\left(i\mathbf{k} \cdot \sum_i \mathbf{r}_i\right) \tag{40}$$

After the above transformation, we define the weights in Fourier space by

$$\tilde{\mathscr{U}}(I, \mathbf{k}) = \sum_{\mathbf{r}} \exp(i\mathbf{k} \cdot \mathbf{r})\, \mathscr{U}(I, \mathbf{r}) \tag{41}$$

From (39) we get the bound on $\tilde{\mathscr{U}}(I, \mathbf{k})$

$$\begin{aligned} |\tilde{\mathscr{U}}(I, \mathbf{k})| &\leqslant \sum_{\mathbf{r}} c \exp\{-[|\mathbf{r}|/2 + 2(d-1)\,|I|]\,\eta\} \\ &\leqslant O(\exp[-2(d-1)\,|I|\,\eta]) \end{aligned} \tag{42}$$

In terms of our new definitions, after the limits $\beta \to \infty$ and $|\Lambda| \to \infty$ (which limits exist by standard polymer expansion results[5]), (38) becomes

$$f(t) = \exp(-2dt) \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \sum_{\substack{I_1,\ldots,I_m:\\ \text{disjoint}}} \prod_{i=1}^{m} \bar{\mathscr{U}}(I_i, \mathbf{k}) \tag{43}$$

We define

$$Z_\tau(t, \mathbf{k}) \stackrel{\text{def}}{=} \sum_{\substack{I_1,\ldots,I_m:\\ \text{disjoint}}} \prod_{i=1}^{m} \bar{\mathscr{U}}(I_i, \mathbf{k}) \tag{44}$$

Based on (42) we can assert the existence of an expansion for the logarithm of $Z_\tau(t, \mathbf{k})$, and thus

$$f(t) = \exp(-2dt) \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \times \exp\left[\frac{tf_\infty(\mathbf{k})}{\tau} + f_1(\mathbf{k}) + f_2(t, \mathbf{k})\right] \tag{45}$$

where

$$f_\infty(\mathbf{k}) = \sum_{\substack{I_1,\ldots,I_m:\\ \text{left edge of } \bigcup_i I_i = 0}} V^C(I_1,\ldots, I_m) \prod_i \lim_{t \to \infty} \bar{\mathscr{U}}(I_i, \mathbf{k}) \tag{46}$$

and the other terms in the exponent of (45) are due to boundary effects and finite-t effects (i.e., from intervals which include one or two endpoints, respectively). Both f_1 and f_2 are uniformly bounded in $\mathbf{k}$ from (42). In addition, f_1 has no t dependence, and f_2 is bounded by $O(e^{-\rho t})$ for $\rho = 2(d-1)\eta/\tau$. At this point in the calculations, the Ising model will have a similar result up to differences in the functions f_∞, f_1, and f_2.

2.4. Analysis of the Expansion

To get results about the spectrum of H we must make a change of variables in the integral in (45). If we let $g(t)$ be the integral

$$\frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \exp\left[\frac{tf_\infty(\mathbf{k})}{\tau} + f_1(\mathbf{k}) + f_2(t, \mathbf{k})\right] \tag{47}$$

we claim that the integral can be written as

$$g(t) = \int_{-\infty}^{\infty} \exp(-tE)\, d\mu(E) \tag{48}$$

where $d\mu$ is a measure on $[-M, \infty)$ for $0<M<\infty$ to be chosen later [the value of M will be at most $2d$—see (45)]. To prove the existence of the proposed measure we will apply the Riesz–Markov theorem. Let $\mathscr{X}$ be the one-point compactification of $[-M, \infty)$. By verifying the hypotheses of the Riesz–Markov theorem for a set of functions whose closure is the set of all continuous functions on $\mathscr{X}$, we show that the hypotheses hold for all continuous functions on $\mathscr{X}$ and then conclude the existence of a unique measure $d\mu$.

To the following set of functions we apply the Stone–Weierstrass theorem to show that its closure is the set of all continuous functions on $\mathscr{X}$. Consider the set of functions in the subalgebra of $C_{\mathbf{R}}(\mathscr{X})$ given by

$$B=\left\{\sum_{i=1}^{N} c_i \exp(-t_i E) \mid t_i \geqslant 0, c_i \in \mathbf{R}, N \in \mathbf{Z}^+\right\} \tag{49}$$

It is easy to verify that functions in B contain 1, and separate points in $\mathscr{X}$. Therefore by the Stone–Weierstrass theorem $\bar{B}=C_{\mathbf{R}}(\mathscr{X})$ (where the bar indicates the closure of the set). Now define the following functional on B, for $b=\sum_i c_i \exp(-t_i E)$:

$$\begin{aligned}\phi(b)&=\phi\left(\sum_i c_i \exp(-t_i E)\right)\\&\stackrel{\text{def}}{=}\sum_i c_i g(t_i)\end{aligned} \tag{50}$$

One can easily check that the definition of $\phi(b)$ is well defined for all $b \in \bar{B}$ [and thus for all functions in $C_{\mathbf{R}}(\mathscr{X})$]. The definition (50) implies the following:

$$\begin{aligned}\phi(b)&=\sum_i c_i \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \exp\left[\frac{t f_\infty(\mathbf{k})}{\tau}+f_1(\mathbf{k})+f_2(t_i, \mathbf{k})\right]\\&=\sum_i c_i \lim_{\alpha \to 0} \sum_E \exp(-t_i E) \int_{\{(-f_\infty/\tau)^{-1}([E, E+\alpha))\}}\\&\quad\times \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \exp[f_1(\mathbf{k})+f_2(t_i, \mathbf{k})]\\&\stackrel{\text{def}}{=}\sum_i c_i \int_{-\infty}^{\infty} \exp(-t_i E)\, d\mu_{t_i}(E)\end{aligned} \tag{51}$$

where in the second equality E ranges over the set $\{-M, -M+\alpha, -M+2\alpha, \ldots\}$. It is easy to verify that, as defined above, $d\mu_t$ is an integration measure for which $\mu_t(\mathscr{X})$ is uniformly bounded in t (recall the bounds on f_1 and f_∞), and which gives the point at infinity zero mass. To show that the functional $\phi(\cdot)$ is linear is trivial, and to show that it is bounded is also easily done:

$$\begin{aligned}|\phi(b)| &\leqslant \int_{-\infty}^{\infty} \left|\sum_i c_i \exp(-t_i E)\right| d\mu_t(E) \\ &\leqslant \|b\|_\infty \int_{-\infty}^{\infty} d\mu_t(E) \\ &= \text{const} \cdot \|b\|_\infty \end{aligned} \tag{52}$$

where the constant does not depend on t. From the last line in (51) it is also clear that $\phi(\cdot)$ is a positive functional. Therefore, by the Riesz–Markov theorem there is a unique Baire measure on $\mathscr{X}$ such that (48) holds true. We leave the determination of the support of $d\mu$ (which will determine the value of M) until later.

Next we show that the measure in (48) has a continuous part. We define $h(t)$ by

$$\begin{aligned} h(t) &\stackrel{\text{def}}{=} \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \exp\left[\frac{tf_\infty(k)}{\tau} + f_1(\mathbf{k})\right] \\ &= \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \exp\left[\frac{tf_\infty(\mathbf{k})}{\tau} + f_1(\mathbf{k})\right] \\ &= \lim_{\alpha \to 0} \sum_E \exp(-t_i E) \int_{\{(-f_\infty/\tau)^{-1}([E,\, E+\alpha))\}} \\ &\quad \times \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \exp[f_1(\mathbf{k})] \\ &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \exp(-t_i E)\, d\tilde{\mu}(E) \end{aligned} \tag{53}$$

The measure $d\tilde{\mu}$ is continuous as long as $f_\infty(\mathbf{k})$ is not constant on any region of nonzero area. We will next show that this measure is equal to the measure $d\mu$ on the E interval $[-M, \rho-\delta)$ for some δ which depends only on d and η, for which $0<\delta<\rho$ for large enough η. The bounds on f_1 and

f_2 are $|f_1(\mathbf{k})| \leqslant c_1$ and $|f_2(t, \mathbf{k})| \leqslant c_2 \exp(-\rho t)$. Thus, asymptotically as $t \to \infty$,

$$|g(t) - h(t)| \leqslant \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \exp\left[\frac{tf_\infty(\mathbf{k})}{\tau} + f_1(\mathbf{k})\right] \times \left| f_2(t, \mathbf{k}) + \frac{f_2(t, \mathbf{k})^2}{2} + \cdots \right|$$

$$\leqslant c_2 \exp(-\rho t) \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} dk_1 \cdots \int_{-\pi}^{\pi} dk_d \times \exp\left[\frac{tf_\infty(\mathbf{k})}{\tau} + f_1(\mathbf{k})\right] [1 + O(\exp(-\rho t))] \qquad (54)$$

Now assuming that the remaining integral above does not grow faster than $\exp(\delta t)$ for some $\delta < \rho$, we have shown that the measures appearing in $g(t)$ and $h(t)$ are identical for energies in the interval $[-M, \rho - \delta)$. Therefore, if the measure $d\tilde{\mu}$ is continuous on some subinterval of $[-M, \rho - \delta)$, the measure $d\mu$ is also continuous on that subinterval.

In order to prove the claims before and after (54) we will first show that the integral in (54) has an asymptotic expansion which does not grow too fast as t increases. To do the integral for large t we will use the method of steepest descent, which requires that we investigate the dependence of (46) on $\mathbf{k}$. In turn we must consider (41). The values of $\mathbf{r}$ for which $\mathscr{U}(I, \mathbf{r})$ can be nonzero are restricted to even values of $|\mathbf{r}|$ (where $|\cdot|$ is the l_1 distance). The $\mathbf{r} = 0$ terms in the sum (41) will have no $\mathbf{k}$ dependence, and can be taken out of the integral in (45). The largest (in absolute value) $\mathbf{k}$-dependent terms in (41) will be those with $|\mathbf{r}| = 2$ [recall the bound (39)]. Similarly, the largest terms in (46) have only one interval I of length τ. Therefore,

$$f_\infty(\mathbf{k}) = \mathscr{U}([0, \tau), 0) + \sum_{\mathbf{r}: |\mathbf{r}| = 2} \exp(i\mathbf{k} \cdot \mathbf{r}) \, \mathscr{U}([0, \tau), \mathbf{r}) + O(\exp(-2d\eta)) \qquad (55)$$

We must focus on the second term in the preceding sum, or more specifically, the largest terms in $\mathscr{U}([0, \tau), \mathbf{r})$ for $|\mathbf{r}| = 2$. There are basically two types of excitations which can appear when $|\mathbf{r}| = 2$: either $\mathbf{r}$ lies entirely along the direction of a unit vector in Λ, or $\mathbf{r}$ is the sum of two orthogonal unit vectors. For $d = 2$, Fig. 3 shows examples of these two types of excitations. Both types of excitations have identical, positive weights (which we will denote by G). We can then write the sum as

$$\sum_{\mathbf{r}:|\mathbf{r}|=2} \exp(i\mathbf{k}\cdot\mathbf{r})\,\mathscr{U}([0,\tau),\mathbf{r})$$
$$= 2G\left\{\sum_{i=1}^{d}\cos(2k_i) + \sum_{i<j=1}^{d}[\cos(k_i+k_j)+\cos(k_i-k_j)]\right\}$$
$$= 2G\left[\sum_{i=1}^{d}\cos(2k_i) + \sum_{i<j=1}^{d}(2\cos k_i\cos k_j)\right] \tag{56}$$

Therefore the right-hand side of (53) is asymptotically (as $t\to\infty$) given by

$$\exp\{\mathscr{U}([0,\tau),0)\,t/\tau\}\exp(2Gd^2t/\tau)$$
$$\times\prod_{j=1}^{d}\int_{-\infty}^{\infty}dk_j\exp\{-t[ck_j^2+O(|\mathbf{k}|^4)]\} \tag{57}$$

where c is a positive number depending only upon d. And after a change of variables this becomes

$$\exp\left[\frac{\mathscr{U}([0,\tau),0)\,t}{\tau}\right]\exp\left(\frac{2Gd^2t}{\tau}\right)$$
$$\times\prod_{j=1}^{d}\int_{0}^{\infty}\exp(-tE_j)\left[\frac{1}{2(cE_j)^{1/2}}+O\left(\frac{1}{t}\right)\right]dE_j \tag{58}$$

Evaluating the integral above shows a leading-order t dependence of $t^{-d/2}$. If we let $\delta=(\mathscr{U}([0,\tau),0)+2Gd^2)/\tau$ (which depends only on the parameters d, η, and τ), then δ can be made less than ρ for large enough η. Note that the continuity of the measure $d\tilde{\mu}$ near the origin has also been established since we have found an expression for $f_\infty(\mathbf{k})$ which is not constant near the origin. Thus the claims preceding and following (54) have been proved, and $d\mu$ is therefore absolutely continuous at least for an

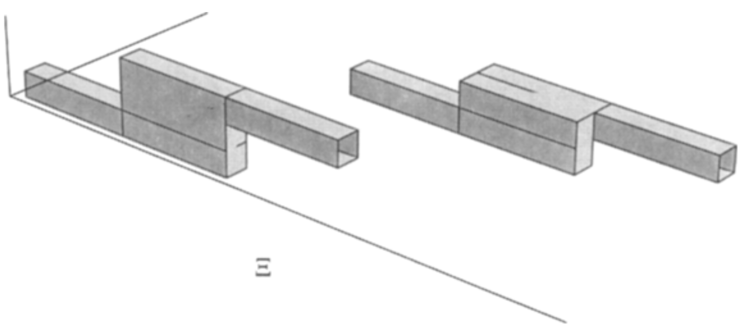

Fig. 3. Examples of the two types of excitations for $|\mathbf{r}|=2$ with the largest weights ($d=2$).

interval near the origin. The location and size of this interval are determined below.

With only a little work we can find the width of the gap in this portion of the spectrum above the ground-state energy and show that the continuous part of the spectrum has width of $O(\varepsilon)$. Because $h(t)$ and $g(t)$ have identical measures out to order $1-\delta$ [recall that $\delta \sim O(e^{-\eta}/\tau)$], we may use $h(t)$ rather than $g(t)$ to investigate the measure within $O(\varepsilon)$ of 0. From (53) we find that the support of $d\tilde{\mu}$ is the image of the function $-f_{\infty}/\tau$. Referring to (55) and (56), we find that f_{∞} has its absolute maximum value at $\mathbf{k}=0$. Therefore the minimum value of E in the support of $d\tilde{\mu}$ (and also $d\mu$) is $-f_{\infty}(0)/\tau$ [and the value of M which was left undetermined above should be at least $f_{\infty}(0)/\tau$]. The width of the support of $d\tilde{\mu}$ is found by identifying the minimum value of $f_{\infty}(\mathbf{k})/\tau$. Again using the same two equations, we find that the minimum is $\{\mathscr{U}([0,\tau),0)-2Gd^2\}/\tau$, which is $O(\delta)$. Because the maximum is also $O(\delta)$, the support of $d\tilde{\mu}$ has width of $O(\delta)$, which is $O(\varepsilon)$. The arguments at the beginning of this paragraph then allow us to conclude that the continuous part of the measure $d\mu$ has a width of $O(\varepsilon)$.

Lastly, we prove the analyticity of the gap in ε. We have found that if we take the ground-state energy to be 0, then the lower edge of the continuous spectrum that we have found has energy $2d-f_{\infty}(0)/\tau$. The definition of $f_{\infty}(\mathbf{k})$, (46), is an absolutely convergent series, and is therefore analytic in ε if each of its terms is an analytic function of ε. We will therefore show that $\bar{\mathscr{U}}(I,\mathbf{k})$ is an analytic function of ε. Tracing back definitions, we find that it is sufficient to show that the weights of polymers are analytic functions of ε. Since the weights of polymers were simply products whose factors were restricted to ε/N and $\exp(-1/N)$, the weights of polymers are indeed analytic functions of ε.

3. CONCLUSIONS

Using the observable σ_0^x, we have shown that the spectrum of the infinite-volume ground states of the two Hamiltonians considered here contain a continuous component directly above the mass gap. We must note that to get the full spectrum we would have to carry out our calculations for all observables and all t. For large values of t one can concieve how the calculation would fare using an observable such as $\sigma_0^x\sigma_{\mathbf{r}}^x$. For this observable, there are several possible conformations of the tubes which are of lowest energy. If the l_1 distance between the two sites at time 0 is odd, the two tubes may disappear altogether in the interior; as soon as the two tubes are adjacent to each other, the $\sigma_i^+\sigma_j^+$ (or $\sigma_i^-\sigma_j^-$) operator may end both tubes. On the other hand, if the sites at time 0 are an even l_1

distance apart, then the two tubes will never touch. Although this "two-site" observable is the next simplest observable to consider, it introduces new difficulties. We must be careful since there is the potential for the appearance of two-magnon bound states.[11] If bound states occur, there will be a discrete component to the spectrum.

We can also consider how our calculation would proceed using observables which are products of σ^x operators for more than two sites. The essential difference in these "multiple-site" observables from the two- or one-site observables is that there will be many possible conformations of tubes of lowest energy, and the different conformations will occur with different entropies. For example, in $d=2$ with a three-site observable, in a view perpendicular to the Ξ axis, the three sites can lie in an "I" (i.e., collinear) or an "L" (i.e., on the vertices of a right triangle) shape. The "I" conformation can appear in two distinct orientations, while the "L" conformation can appear in four distinct orientations. All of the "I" and "L" conformations will have the same energy, but due to entropy effects, will have different free energies. This free energy difference makes the derivation of the expansion fail.

All of the foregoing discussion shows that the spectrum in the ground state will still have many more features than we have been able to reveal using our expansion technique. However, the features of the spectrum which we have found using the single-site observable are nonetheless valid. While we cannot rule out the possibility of discrete or singular components of the spectrum, we can conclude that a continuous band near the ground state is a feature of the spectrum of both the spin-1/2 anisotropic Heisenberg model in dimensions greater than or equal to two, and the Ising model in a strong transverse field in dimensions greater than or equal to one.

ACKNOWLEDGMENTS

The author is indebted to T. Kennedy, whose assistance in this work was essential. The author was partially supported in this work by NSF grants DMS-8902248 and DMS-9103621.

REFERENCES

1. C. Albanese, On the spectrum of the Heisenberg Hamiltonian, *J. Stat. Phys.* **55**:297 (1989).
2. O. Bratelli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I* (Springer-Verlag, New York, 1979).
3. O. Bratelli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II* (Springer-Verlag, New York, 1981).

4. J. Bricmont and J. Fröhlich, Statistical mechanical methods in particle structure analysis of lattice field theories, *Commun. Math. Phys.* **98**:553 (1985).
5. D. Brydges, A short course on cluster expansions, in *Critical Phenomena, Random Systems, Gauge Theories*, K. Osterwalder and R. Stora, eds. (Elsevier, 1986).
6. J. Ginibre, Existence of phase transitions for quantum lattice systems, *Commun. Math. Phys.* **14**:205 (1969).
7. T. Kennedy, Ornstein–Zernicke decay in the ground state of the quantum Ising model in a strong transverse field, *Commun. Math. Phys.* **137**:599 (1991).
8. T. Kennedy and H. Tasaki, Hidden symmetry breaking and the Haldane phase in $S=1$ quantum spin chains, *Commun. Math. Phys.* (1992).
9. J. R. Kirkwood and L. E. Thomas, Expansions and phase transitions for the ground state of quantum Ising lattice systems, *Commun. Math. Phys.* **88**:569 (1983).
10. R. Kotecky and D. Preiss, Cluster expansions for abstract polymer models, *Commun. Math. Phys.* **103**:491 (1986).
11. D. C. Mattis, *The Theory of Magnetism* (Springer-Verlag, New York, 1987).
12. A. J. Schorr, Antiferromagnetic quantum spin chains with an anisotropic coupling, Senior Thesis, Princeton University, Princeton, New Jersey (1987).
13. L. E. Thomas and Z. Yin, Low temperature expansions for the Gibbs states of weakly interacting quantum Ising lattice systems, *Commun. Math. Phys.* **91**:405 (1983).